CMPUT 397 Reinforcement Learning:

Probabilities & Expectations

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Probabilities and intelligent systems

- ✓ Probability is a measure of uncertainty
- ✓ An intelligent system maximizes its “chances” of success
- ✓ Intelligent systems create a favorable future
- ✓ Probabilities and expectations are tools for reasoning about uncertain future events

Let's take the example of rolling a dice

- ✓ We say the probability of observing 3 is 1/6
- ✓ How to express it mathematically?
- ✓ Rolling a dice is, an **experiment**, a repeatable process

  with different possible results/outcomes
- ✓ One **outcome** is 3. Outcomes are mutually exclusive
- ✓ The set of all outcomes is called a **sample space**: { 1, 2, 3, 4, 5, 6 }
- ✓ An **event** is a set of outcomes. The event of observing 4 or more: {4, 5, 6}
- ✓ Define ***P*** as a function mapping from events to probabilities: $P(3) = 1/6$

Probability axioms

- **Non-negativity:** A probability is always non-negative

$$0 \leq P(A), \text{ for all } A$$

what kind of object is this?

- **Additivity:** If $A \cap B = \{\ \}$, then $P(A \cup B) = P(A) + P(B)$
- **Unit measure:** $P(\Omega) = 1$, where Ω is the sample space
- What is the probability of observing 4 or more?
- $P(\{4, 5, 6\}) = P(4) + P(5) + P(6) = 3/6 = 1/2$

Random variables

- ✓ Random variables are a convenient way to express events
- ✓ A Random variable is a function mapping from outcomes to real values
- ✓ For coin-tossing experiment: it can be X(head) = 1 and X(tail) = -1
- ✓ For outcomes of dice-rolling experiment: X(a) = a
- ✓ It allows succinct expressions for events such as $[X \geq 4]$

  which stands for $\{ \omega \in \Omega : X(\omega) \geq 4 \} = \{ 4, 5, 6 \}$

Random variables: example

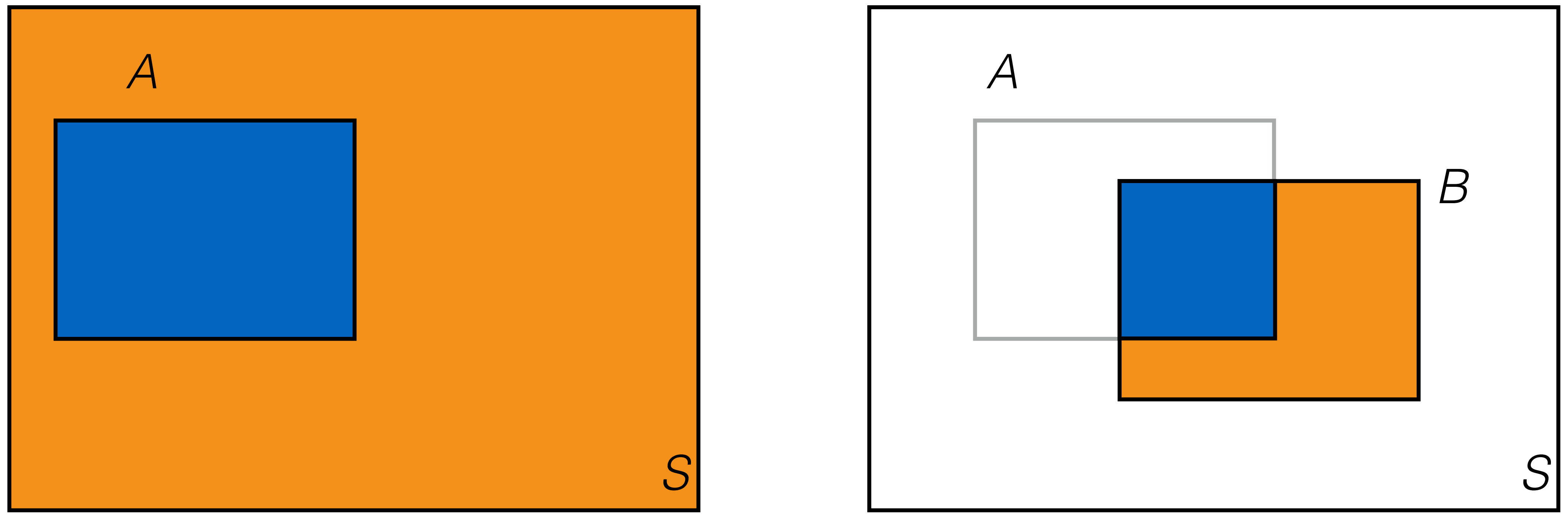
- ✓ If we roll two dices, what is the probability of the sum being more than 2?
- ✓ Sample space: { (1,1), …, (1,6), (2,1), …, (2,6), …, (6,1), …, (6,6) }
- ✓ We can define a random variable X standing for the sum
- ✓ Then the event of "the sum being more than 2" can be written as $[X > 2]$
- ✓ Then $1 = P(\Omega) = P([X = 2] \cup [X > 2]) = P(X = 2) + P(X > 2)$

Conditional probabilities

- ✓ A conditional probability is a measure of an uncertain event when we know that another event has occurred

- ✓ In the single dice-rolling experiment, if the sum is below 4, what is the probability that the value is more than 2

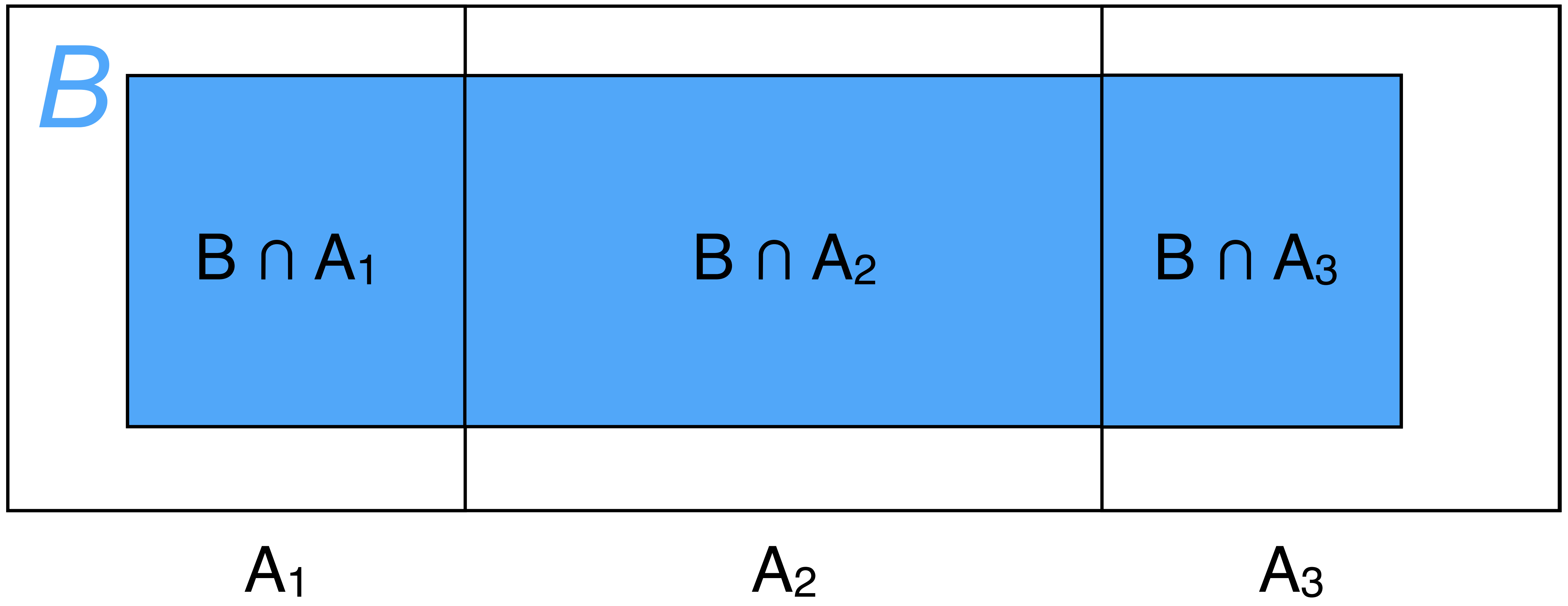
- ✓ Definition: $P(A \mid B) = P(A \cap B) \,/\, P(B) \neq P(A)$



Conditional probabilities: example

- In the single dice-rolling experiment, if the sum is below 4, what is the probability that the value is more than 2
- $P([Z > 2 \mid Z < 4])$
- $= P([Z > 2] \cap [Z < 4]) / P([Z < 4])$
- $= P([Z = 3]) / P([Z < 4])$
- $= (1/6) / (1/2) = 1/3$

Low of total probabilities



$$A_i \cap A_j = \phi,\ i \neq j, \qquad \cup_i A_i = \Omega$$

$$P(B) = \sum_k P(B \cap A_k)$$

$$= \sum_k P(B \mid A_k)\, P(A_k)$$

Expectations & conditional expectations

- ✓ An expected value of a random variable is a weighted average of possible outcomes, where the weights are the probabilities of those outcomes

$$\mathbf{E}[X] = \sum_{x \in \mathcal{X}} x \, P(X=x)$$

- ✓ An expected value of a random variable conditional on another event is a weighted average of possible outcomes, where the weights are the conditional probabilities of those outcomes given the event

$$\mathbf{E}[X \mid Y=y\,] = \sum_{x \in \mathcal{X}} x \, P(X=x \mid Y=y\,)$$

- ✓ Expectation conditional on a random variable $\mathbf{E}[X \mid Y\,]$ itself is a random variable, which is a function of another random variable Y

Properties of expectations

- ✓ Linearity: $\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$
- ✓ Linearity: $\mathbf{E}[aX] = a\mathbf{E}[X]$
- ✓ Non-multiplicativity: $\mathbf{E}[XY] \neq \mathbf{E}[X]\,\mathbf{E}[Y]$
- ✓ Law of the unconscious statistician: $\mathbf{E}[\,g(X)\,] = \sum_{x \in \mathcal{X}} g(x)\, P(X=x)$

Expectations: example

- ✓ In the double dice-rolling experiment, What is the expected value of the sum of the two dice?

Expectations: example

- ✓ Show that $\mathbf{E}[X] = \mathbf{E}[\,\mathbf{E}[X \mid Y]\,]$